THE INEQUALITIES THAT DETERMINE THE BARGAINING SET $\mathscr{M}_1^{(i)}$

BY

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ABSTRACT

It is well-known that the payoffs of the various bargaining sets of a cooperative n-person game are finite unions of closed convex polyhedra. In this paper, the system of inequalities that determines these polyhedra for the bargaining set $\mathscr{M}_1^{(i)}$ is expressed in explicit form.

It turns out that this system also expresses the condition that certain games, derived from the original game and from the potential payoffs, have full-dimensional cores.

1. **Introduction**. The basic papers dealing with the bargaining set $\mathscr{M}_1^{(i)}$ are [4] by M. Davis and the author, and [9] by B. Peleg. (See also [6], and [5] by M. Davis and the author.) For intuitive justification of the bargaining set as a solution concept, the reader is referred to [1] by R. J. Aumann and the author, where other bargaining sets are described. It is shown in [1] that one of the bargaining sets can be represented as a solution of a finite set of linear weak inequalities connected by the words "and" and "or." A similar proof holds for $\mathscr{M}_1^{(i)}$.

The purpose of the present paper is to provide such a system for $\mathscr{M}_1^{(i)}$ in explicit form.

2. **The system whose solution is the bargaining set $\mathscr{M}_1^{(i)}$**. Let $(N; v)$ be a cooperative n-person game in characteristic function form, where $N = \{1, 2, \cdots, n\}$ is its set of players and its characteristic function v is assumed to satisfy

$$(2.1) \qquad v(S) \geqq \sum_{i \in S} v(\{i\})$$

for each coalition (i.e., non-empty subset of N).

Let $\mathscr{B} \equiv \{B_1, B_2, \cdots, B_m\}$ be a *coalition-structure*; i.e., a partition of N into m disjoint coalitions.

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An *individually rational payoff configuration* (i.r.p.c.) is the pair

$$(x;\mathscr{B}) \equiv (x_1, x_2, \cdots, x_n;\ B_1, B_2, \cdots, B_m),$$

where $x \equiv (x_1, x_2, \cdots, x_n)$—called the *payoff-vector*—is an n-tuple of real numbers satisfying

$$\sum_{i \in B_j} x_i = v(B_j), \qquad j = 1, 2, \cdots, m. \tag{2.2}$$

$$x_i \geqq v(\{i\}), \qquad i = 1, 2, \cdots, n \text{ (individual rationality)}. \tag{2.3}$$

Denote by $\mathscr{T}_{k,l}$ the set of coalitions which contain a player k and do not contain a player l.

Let $(x;\mathscr{B})$ be an i.r.p.c. for a game $(N;v)$ and let k and l be two distinct players who belong to the *same* coalition in $\mathscr{B}$. An objection of k against l, with respect to $(x;\mathscr{B})$ is a pair(1) $(\hat{y};C)$, where $C \in \mathscr{T}_{k,l}$ and $\hat{y} \equiv (y_i)_{i \in C}$ is a c-tuple of real numbers satisfying

$$\sum_{i \in C} y_i = v(C), \tag{2.4}$$

$$y_i > x_i, \qquad i \in C. \tag{2.5}$$

(c denotes the number of players in C.)

Let $(x;\mathscr{B})$ be an i.r.p.c. for a game $(N;v)$ and let $(\hat{y};C)$ be an objection of a player k against a player l, (k, l being distinct players in the same coalition in $\mathscr{B}$). A *counter-objection* to the above objection is a pair $(\hat{z};D)$, where $D \in \mathscr{T}_{l,k}$ and $\hat{z} \equiv (z_i)_{i \in D}$ is a d-tuple of real numbers satisfying

$$\sum_{i \in D} z_i = v(D), \tag{2.6}$$

$$z_i \geqq x_i, \qquad i \in D, \tag{2.7}$$

$$z_i \geqq y_i, \qquad i \in C \cap D. \tag{2.8}$$

d (denotes the number of players in D.)

We say that k can object against l by *using* the coalition C if an objection $(\hat{y};C)$ of k against l exists. For such an objection we say that l can counter-object by using the coalition D if a counter-objection $(\hat{z};D)$ exists. We say that k has a *justified objection* against l (with respect to an i.r.p.c. $(x;\mathscr{B})$), if there is an objection of k against l which cannot be countered.

The *bargaining set* $\mathscr{M}_1^{(i)}$ is the set of all i.r.p.c.'s with respect to which no player

(1) The ^ symbol indicates that the coordinates of the payoff vector are restricted to a coalition.

has a justified objection against another player (who belongs to the same coalition in the coalition structure).

Let $(x;\mathscr{B})$ be an i.r.p.c. for a game $(N;v)$. For each coalition S, we call

$$e(S,x) \equiv v(S) - \sum_{i\in S} x_i \tag{2.9}$$

the *excess* of S with respect to $(x;\mathscr{B})$. Obviously, $e(B_j) = 0, j = 1,2,\cdots,m$.

LEMMA 2.1. *Let $(x;\mathscr{B})$ be an i.r.p.c. for a game $(N;v)$. Let k and l be two distinct players in a coalition of $\mathscr{B}$. Let C be a coalition in $\mathscr{T}_{k,l}$. In order that k has an objection against l by using the coalition C, it is necessary and sufficient that $e(C,x) > 0$.*

The proof is straightforward. Note that a single-person coalition can never be used for an objection.

In order to find a criterion for k having a *justified objection* against l by using C, it is convenient to construct the $(C;k,l;x)$-game:

DEFINITION 2.2. Let $(x;\mathscr{B})$ be an i.r.p.c. and let (k,l) be an ordered pair of players in a coalition of $\mathscr{B}$. Let C be a coalition in $\mathscr{T}_{k,l}$ which contains at least two members. The *$(C;k,l;x)$-game* is a game $(C-\{k\};v_C)$ over the set of players $C-\{k\}$, whose characteristic function v_C is defined by

$$v_{C,k,l}(S,x) \equiv v_C(S) = \text{Max}\,(0, \text{Max}_{\substack{D\in\mathscr{T}_{l,k}\\ D\cap C=S}} e(D,x)), \tag{2.10}$$

for each coalition S contained in $C-\{k\}$.

Thus, the value of each coalition in $C-\{k\}$ is either 0 or the most that player l can pay the members of S without resorting to the consent of k, whichever is the greatest.

LEMMA 2.3. *Let $(x;\mathscr{B})$ be an i.r.p.c. in a game $(N;v)$ and let k and l be two distinct players in a coalition of $\mathscr{B}$. Let C be a coalition in $\mathscr{T}_{k,l}$ which contains at least two players. Player k has a justified objection against player l (with respect to $(x;\mathscr{B})$) by using the multi-person coalition C if and only if the following conditions are satisfied:*

(i) *$e(D,x) < 0$ whenever $D\in\mathscr{T}_{l,k}$ and $D\cap C=\varnothing$;*

(ii) *There exists an r-tuple of real numbers $\hat{t}\equiv(t_i)_{i\in C-\{k\}}$, where r is the number of players in $C-\{k\}$, such that*

$$\sum_{i\in C-\{k\}} t_i = e(C\ x) \tag{2.11}$$

and for each coalition S in the $(C;k,l;x)$-game

$$(2.12)\qquad \sum_{i \in S} t_i > v_C(S);\quad S \subset C - \{k\}.$$

Proof. A. If $(\hat{y}; C)$ is an objection of k against l, with respect to $(x; \mathscr{B})$ which cannot be countered, set $t_i = y_i - x_i + (y_k - x_k)/r$, $i \in C - \{k\}$. It follows from (2.4) and (2.9) that (2.11) is satisfied. By (2.5), $t_i > 0$, $i \in C - \{k\}$. Suppose that (2.12) were not satisfied for a coalition S_0, then $v_C(S_0) > 0$ and a coalition D_0 would exist in $\mathscr{T}_{l,k}$ such that $S_0 = D_0 \cap C \neq \emptyset$ and

$$(2.13)\qquad t_0 \equiv \sum_{i \in S_0} t_i \leqq v_C(S_0) = e(D_0, x).$$

Therefore, l could counter-object by $(\hat{z}; D_0)$, where $z_i = t_i + x_i$ for $i \in S_0$, $z_l = e(D_0, x) - t_0 + x_l$, $z_i = x_i$ for $i \in D_0 - S_0 - \{l\}$. Indeed, (2.6)–(2.8) are then satisfied for $D = D_0$. This contradiction shows that (2.12), and therefore (ii), are satisfied. If (i) were not satisfied, then there would exist a coalition D_1 in $\mathscr{T}_{l,k}$ such that $D_1 \cap C = \emptyset$ and $e(D_1, x) \geqq 0$. Obviously, l could then counter-object by using D_1. This contradiction shows that (i) is, in fact, satisfied.

B. Suppose (i) and (ii) are satisfied. Set

$$(2.14)\qquad \delta \equiv \operatorname*{Min}_{S \subset C - \{k\}} \left(\sum_{i \in S} t_i - v_C(S) \right).$$

By (2.12), $\delta > 0$. We shall show that $(\hat{y}; C)$, where $y_k = x_k + \delta/2$, $y_i = x_i + t_i$ whenever $i \in C - \{k\} - \{p\}$, $y_p = x_p + t_p - \delta/2$, p being a particular player in $C - \{k\}$, is a justified objection. Indeed, (2.4) is satisfied. Apply (2.14) to the single-person coalitions in $C - \{k\}$ and observe that $v_C \geqq 0$ (see (2.10)). It follows that $t_i > \delta/2 > 0$ whenever $i \in C - \{k\}$.

Thus, (2.5) is also satisfied and $(\hat{y}; C)$ is an objection. Suppose that l has a counter-objection $(\hat{z}; D_2)$. If $D_2 \cap C = \emptyset$ then (2.5)–(2.9) imply $e(D_2, x) \geqq 0$, contrary to (i). If $S_2 \equiv D_2 \cap C \neq \emptyset$ then S_2 is coalition in $C - \{k\}$ and it follows from (2.9) and (2.10) that

$$(2.15)\qquad v_C(S_2) \geqq e(D_2, x) = v(D_2) - \sum_{i \in D_2} x_i.$$

By (2.6)–(2.8), however, $v(D_2) = \sum_{i \in D_2} z_i \geqq \sum_{i \in S_2} y_i + \sum_{i \in D_2 - S_2} x_i \geqq \sum_{i \in S_2} t_i + \sum_{i \in D_2} x_i - \delta/2$. Thus, $\sum_{i \in S_2} t_i - v_C(S_2) \leqq \delta/2$, contrary to (2.14). The contradiction shows that $(\hat{y}; C)$ is, in fact, a justified objection.

Condition (ii) resembles the condition that a certain game has a full dimensional core. Indeed, let $(C - \{k\}; v_C^*)$ be a game where

$$(2.16)\qquad v_C^*(S) = \begin{cases} v_C(S) & \text{whenever } S \text{ is a proper subset of } C - \{k\} \\ e(C, x) & \text{if } S = C - \{k\}. \end{cases}$$

Condition (ii), with (2.12) being restricted to proper subsets of $C - \{k\}$, states that $(C - \{k\}; v_C^*)$ has a full dimensional core. We can, therefore, use the results of O. N. Bondareva [3] and L. S. Shapley [10] in order to state condition (ii) in terms of $v_C(S)$ and $e(C,x)$ alone:

DEFINITION 2.4. (L. S. Shapley). Let T be a non-empty finite set of players. A collection $\mathscr{S} \equiv \{S_1, S_2, \cdots, S_q\}$ of non-empty subsets(2) of T is called *balanced* if there exist positive constants $\gamma_1, \gamma_2, \cdots, \gamma_q$, such that

$$\sum_{j|i \in S_j} \gamma_j = 1, \text{ all } i \text{ in } T. \tag{2.17}$$

The coefficients satisfying (2.17) are called the *weights* for $\mathscr{S}$.

DEFINITION 2.5. (L. S. SHAPLEY). A balanced collection is called *minimal*, if no proper subcollection is balanced(3).

It is known (see [3], [10]) that the weight vector is unique if and only if $\mathscr{S}$ is a minimal balanced collection.

LEMMA 2.6. *Condition* (ii) *of Lemma* 2.3 *holds if and only if for each minimal balanced collection* $\mathscr{S} \equiv \{S_1, S_2, \cdots, S_q\}$ *for* $T = C - \{k\}$,

$$e(C,x) > \sum_{j=1}^{q} \gamma_j v_C(S_j), \tag{2.18}$$

where $(\gamma_1, \gamma_2, \cdots, \gamma_q)$ *is the weight vector for* $\mathscr{S}$.

Proof. Condition (ii) is equivalent to existence of a full dimensional core in the game $(C - \{k\}; v_C^*)$, together with the requirement $e(C,x) > v_C(C - \{k\})$. Since $\{C - \{k\}\}$ is a minimal balanced collection for $C - \{k\}$, it follows that the last requirement is nothing but the application of (2.18) to $\mathscr{S} = \{C - \{k\}\}$. The application of (2.18) to minimal balanced collections other than $\{C - \{k\}\}$ is a necessary and sufficient condition that the game $(C - \{k\}; v_C^*)$ has a full dimensional core (see O. N. Bondareva [3] and L. S. Shapley [10]).

We are now in a position to describe the system of inequalities which determine the bargaining set $\mathscr{M}_1^{(i)}$ of a game:

THEOREM 2.7. *Let* $(N; v)$ *be an n-person cooperative game whose characteristic function satisfies* (2.1). *Let* $\mathscr{B} \equiv \{B_1, B_2, \cdots, B_m\}$ *be a fixed coalition*

(2) L. S. Shapley requires that these subsets will be proper subsets of T. For our purpose it is more convenient to allow T itself to be one of the subsets.

(3) O. N. Bondareva uses similar concepts called "(q-$\mathscr{S}$)-covering of T" and "reduced (q-$\mathscr{S}$)-covering of T."

structure. A necessary and sufficient condition that $(x;\mathscr{B})$ *belongs to the bargaining set* $\mathscr{M}_1^{(i)}$, *where* $x \equiv (x_1, x_2, \cdots, x_n)$, *is:*

(i) $\sum_{i \in B_j} x_i = v(B_j)$, $j = 1, 2, \cdots, m$;

(ii) $x_i \geqq v(\{i\})$, $i = 1, 2, \cdots, n$;

(iii) *for each ordered pair of distinct players* (k, l) *who belong to the same coalition of* $\mathscr{B}$, *and for each coalition* C *in* $\mathscr{T}_{k,l}$ *which contains at least two members, either there exists a coalition* D *in* $\mathscr{T}_{l,k}$ *such that* $D \cap C = \varnothing$ *and* $e(D, x) \geqq 0$ (see (2.9)); *or*

$$e(C, x) \leqq \max_{\mathscr{S} \in \mathbf{R}} \sum_{j | S_j \in \mathscr{S}} \gamma_j(\mathscr{S}) v_C(S_j), \tag{2.19}$$

where $v_C(S_j)$ *is defined by* (2.10), $\mathbf{R}$ *is the set of all minimal balanced collections of* $C - \{k\}$ *and* $\gamma_j(\mathscr{S})$ *is the weight of* S_j *for* $\mathscr{S}$, $S_j \in \mathscr{S}$ (see definitions 2.4 and 2.5).

The proof follows from the Lemmas 2.3 and 2.6.

Theorem 2.7 provides a finite set of linear inequalities connected by the words "and" and "or." Indeed, the "and" connects (i), (ii) and (iii), and also connects the systems which correspond to the various possible ordered triples (k, l, C); whereas "or" connects the various alternatives in (iii), namely, the various possible D's (for a fixed (k, l, C)), and (2.19). In general, (2.19) is not linear, but it is equivalent to the finite set of inequalities: $e(C, x) \leqq \sum_{S_j \in \mathscr{S}^1} \gamma_j(\mathscr{S}^1) v_C(S_j)$ or $\cdots$ or $e(C, x) \leqq \sum_{S_j \in \mathscr{S}^r} \gamma_j(\mathscr{S}^r) v_C(S_j)$, where $\mathbf{R} = \{\mathscr{S}^1, \mathscr{S}^2, \cdots, \mathscr{S}^r\}$. Similarly, each inequality $e(C, x) \leqq \sum_{S_j \in \mathscr{S}^\rho} \gamma_j(\mathscr{S}^\rho) v_C(S_j)$ is equivalent to the finite set of linear inequalities connected by "or," of the type $e(C, x) \leqq \sum_{S_j \in \mathscr{S}^\rho} \gamma_j(\mathscr{S}^\rho) e(D_j^{(\sigma)}, x)$, where the $D_j^{(\sigma)}$, with σ being the changing variable, run over the empty set well as all the sets in $\mathscr{T}_{l,k}$, such that $D_j^{(\sigma)} \cap C = S_j$, and the convention is that $e(\varnothing, x) = 0$.

EXAMPLE 2.8. The following system of inequalities expresses the necessary and sufficient condition that Player 1 has no justified objection against Player 4 with respect to the pair $(x_1, x_2, x_3, x_4;\ \{\{1, 2, 3, 4\}\})$ in the 4-person game $(\{1, 2, 3, 4\}; v)$:

$$x_1 + x_2 + x_3 + x_4 = v(\{1, 2, 3, 4\})$$

and $$x_1 \geqq v(\{1\})$$

and $$x_2 \geqq v(\{2\})$$

and $$x_3 \geqq v(\{3\})$$

and $$x_4 \geqq v(\{4\})$$

and $[x_4 = v(\{4\})$ or $x_3 + x_4 \leqq v(\{3,4\})$ or $x_1 + x_2 \geqq v(\{1,2\})$
or $x_4 - x_1 \leqq v(\{2,4\}) - v(\{1,2\})$ or $x_3 + x_4 - x_1 \leqq v(\{2,3,4\}) - v(\{1,2\})]$

and $[x_4 = v(\{4\})$ or $x_2 + x_4 \leqq v(\{2,4\})$ or $x_1 + x_3 \geqq v(\{1,3\})$
or $x_4 - x_1 \leqq v(\{3,4\}) - v(\{1,3\})$ or $x_2 + x_4 - x_1 \geqq v(\{2,3,4\}) - v(\{1,3\})]$

and $[x_4 = v(\{4\})$ or $x_1 + x_2 + x_3 \geqq v(\{1,2,3\}$ or $x_4 - x_1 \leqq v(\{2,3,4\}) - v(\{1,2,3\})$
or $x_4 - x_1 - x_3 \leqq v(\{2,4\}) - v(\{1,2,3)\}$ or $x_4 - x_1 - x_2 \leqq v(\{3,4\}) - v(\{1,2,3\})$
or $2x_4 - x_1 \leqq v(\{2,4\}) + v(\{3,4\}) - v(\{1,2,3\})]$.

This is the union of 150 convex polyhedra, each of which is the intersection of 8 (possibly coinciding) half spaces.

Taking 12 permutations of the above system, obtained by permuting (1,4) with all the ordered pairs, and connecting the systems by "and," one obtains the necessary and sufficient condition that $(x_1, x_2, x_3, x_4; \{\{1,2,3,4\}\})$ belongs to $\mathscr{M}_1^{(i)}$. Thus, $\mathscr{M}_1^{(i)}$ is represented here as a union of 150^{12} convex polyhedra, each of which is an intersection of (possibly coinciding) 41 half spaces.

Note that in order to check whether a particular payoff (x_1, x_2, x_3, x_4) is in $\mathscr{M}_1^{(i)}$ for the coalition structure $\{\{1,2,3,4\}\}$, only 197 inequalities need be checked.

From some experience with actual computations, however, it appears that most of the 150^{12} polyhedra are perhaps empty and that many are contained in others. The following questions therefore arise:

1. How does one reduce the number of polyhedra that need be computed for a general game?

2. How does one reduce the number of polyhedra that need be computed for a particular game, taking into account the particular properties of the characteristic function?

(See [7] and [8] by B. Peleg and the present author, and [2] by R. J. Aumann, B. Peleg and P. Rabinowitz, where questions of this kind are answered for a representation of the *kernel* of a cooperative game.)

References

1. R. J. Aumann and M. Maschler, *The bargaining set for cooperative games*, Advances in Game Theory, M. Dresher, L. S. Shapley and A.W. Tucker, eds., Annals of Mathematics Studies, No. 52, Princeton University, Princeton, New Jersey, (1964), 443–476.

2. R. J. Aumann, B. Peleg and P. Rabinowitz, *A method for computing the kernel of n-person games*, Mathematics of Computation, **19** (1965), 531–551.

3. O. N. Bondareva, *Some applications of linear programming methods to the theory of cooperative games*, Problemy Kibernetiki **10** (1963), 119–139 (Russian).

4. M. Davis and M. Maschler, *Existence of stable payoff configurations for cooperative games*, Bull. Amer. Math. Soc. **69** (1963), 106–108. A detailed paper with the same title will appear in Studies in Mathematical Economics, Essays in Honor of Oskar Morgenstern, M. Shubik, ed.

5. M. Davis and M. Maschler, *The kernel of a cooperative game*, Naval Research Logistics Quarterly **12** (1965), 223–259.

6. M. Maschler, *n-person games with only* 1, $n-1$, *and n-person permissible coalitions*, J. Math. Analysis and Appl., **6** (1963), 230–256.

7. M. Maschler and B. Peleg, *A characterization, existence proof and dimension bounds for the kernel of a game*, Pacific J. Math. **18**, (1966), 289–328.

8. M. Maschler and B. Peleg, *The structure of the kernel of a cooperative game*, (To appear in *Siam J. on Appl. Math.*)

9. B. Peleg, *Existence theorem for the bargaining set* $\mathcal{M}_1^{(i)}$, Bull. Amer. Math. Soc. **69** (1963), 109–110. A detailed paper with the same title will appear in Studies in Mathematical Economics, Essays in Honor of O. Morgenstern, M. Shubik, ed.

10. Shapley, Lloyd, S. *On balanced sets and cores*, The RAND Corporation, Santa Monica, California, Memorandum RM-4601-PR (June 1965).

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